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THE MECHANICS OF SELF-SIMILAR CRACK GROWTH IN AN ELASTIC POWER-LAW CREEPING MATERIAL

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Abstract—The mechanics of a plane strain Mode I growing crack under transient conditions is investigated using self-similar solutions. For small crack extensions, the evolution of the near tip stress field of Hui and Riedel (HR) is posed as a singular perturbation problem. For small crack extensions, the HR field, the HRR field and the elastic K_I field coexist near the crack tip, one inside the other. The regions of dominance of these fields are estimated. An approximate solution is provided for the singular perturbation problem. The transition time of Riedel and Rice, which is strictly accurate for stationary cracks, remains accurate for growing cracks provided that crack extension is small. The effect of crack growth rate on the small scale yielding assumption is also studied. For fast crack growth, it is shown that creep relaxation can be neglected and results of steady-state analysis can be modified to describe the near tip stress fields. Application of the results to non-self-similar crack growth is also discussed.

1. INTRODUCTION

A common mode of failure of metallic structures is the extension of a single preexisting macroscopic crack. Under creep conditions, this extension occurs by the nucleation, growth and interlinkage of voids and microcracks. The stress concentration near the crack tip promotes the void coalescence. This mode of failure has been reported by many investigators[1–4]. Failure can occur with either a little or a substantial amount of creep deformation in the overall test specimen. In the former case, material remote from the growing crack tip experiences very little creep straining, and the crack growth rate is found to correlate with the stress intensity factor. In the latter case, elastic strains are small compared with the creep strains in the specimen, and crack growth rate is usually found to be governed by C^*. These two limiting behaviors are usually referred to as "K controlled crack growth" and "C^* controlled crack growth", respectively.

The aim of understanding creep crack growth is to provide a quantitative answer to the question, How does the creep crack growth rate depend on the applied loading history? Assuming the material constitutive model governing deformation and the local mechanism of fracture are known, this question is answered by solving equations of continuum mechanics coupled with the equations governing the local crack growth. The nonlinearity of these coupled equations renders the quantitative analysis extremely difficult, even for the simplest loading histories. Part of the mathematical difficulty can be removed by assuming that the local mechanism of fracture does not significantly affect the deformation field. This assumption decouples the equations of mechanics from the equations governing the local crack growth criterion. The resulting stresses obtained from solving the equations of continuum mechanics are then, in general, functionals of the unknown crack growth history, i.e. $\dot{a}(t)$, $t_i \leq t < \infty$, where $\dot{a}(t)$ is the crack growth rate at time t, and t_i is the time of initiation of crack growth. The unknown crack growth history can sometimes then be determined by substituting the deformation field obtained from solving the equations of mechanics into the crack growth criterion. In this paper, the crack growth history will not be forced to be consistent with any particular fracture criterion.

In the continuum mechanics problem, the growth history can be prescribed independent of the load history. The function $\dot{a}(t)$ can therefore be considered as pre-

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scribed but arbitrary. In this paper, we will examine the transient crack growth problem by considering a particular kind of growth history, i.e. we choose the crack growth rate so that the solution of the governing equations is self-similar. The advantage of a similarity solution is that it reduces the number of independent variables in the problem by one. This simplification allows us to gain insight into the time evolution of the near tip stress fields and the far field boundary conditions, thus enabling us to see how the limiting situations of "K controlled crack growth" and "C^* controlled crack growth" are approached in a particular case. This problem will be considered only for the case of an elastic power-law creeping material.

Analyzing the special case of self-similar crack growth may be questioned, as it is unlikely that self-similar crack growth occurs in experiments, and it has, perhaps, little direct relevance to common material tests. The following facts justify our study of the self-similar solution:

a. There exists no transient solution for a growing crack in an elastic power-law creeping material.
b. The self-similar solution can be used to access the accuracy of future numerical work.
c. Certain important features of the self-similar solution may be extended to a wider class of crack growth histories.

There are obvious limitations to the results presented in this work: only sharp cracks in elastic power-law creeping material are considered in the analysis. Other deformation mechanisms (e.g. primary creep, classical plasticity, etc.) have been excluded from this treatment for two reasons. First, mathematical difficulties; second, to a lesser extent, the possibility that these deformation mechanisms may be dominated by power-law creep in some tests. The sharp crack assumption implies that we are modelling fracture caused by failure localized near the crack tip rather than general creep rupture where damage is observed to be smoothly distributed.

The plan of this paper is as follows: In Section 2, we will summarize the constitutive law and some results in the mechanics of creep crack growth. In Section 3, the governing equations of transient crack growth are derived for the case of a plane strain Mode I crack. In Section 4, self-similar solutions of this equation are presented. The physical and mathematical consequences of these solutions will also be interpreted. The extension of these results to the case of arbitrarily prescribed crack growth rate is presented in Section 5. Section 6 summarizes the main results in this paper.

2. CRACK TIP STRESS FIELD

Results on crack tip stress fields will be reviewed in this section. There is no attempt to provide a comprehensive review of current literature. The emphasis is on previous results pertinent to the ensuing discussions. The structure of the presentation closely follows the introductory remarks in Bassani and Vitek[5]. The tensile stress and strain rate relation of an elastic power-law creeping material is

$$\dot{\epsilon} = \dot{\sigma}/E + B\sigma^n, \tag{2.1}$$

where $\dot{\epsilon}$ is the total strain rate, $\dot{\sigma}/E$ the elastic strain rate, $B\sigma^n$ the creep strain rate, E the Young's modulus, B the creep coefficient and n is the creep exponent. For most metals, values of n between 3 and 8 are used to fit experiments. Equation (2.1) includes the short time elastic response and long time creep response given by

$$\dot{\epsilon}^{cp} = B\sigma^n. \tag{2.2}$$

Generalized isotropic stress–strain rate relations which reduce to (2.1) in simple tension are

$$\dot{\epsilon}_{ij} = \frac{1 + \nu}{E}\dot{S}_{ij} + \frac{1 - 2\nu}{3E}\dot{\sigma}_{pp}\,\delta_{ij} + \tfrac{3}{2}B\sigma_e^{n-1}S_{ij}, \tag{2.3}$$

where ν is the Poisson ratio, S_{ij} is the Cartesian component of the deviatoric stress tensor S, and σ_e is the equivalent stress. The crack is assumed to be two dimensional and lying in the plane $y = 0$. The stress field is described by means of a polar coordinate system (r, θ) attached to the crack tip. The ray $\theta = 0$ is identical to the positive x-axis. We will first consider results for stationary cracks, followed by results for growing cracks.

Stationary cracks

Riedel and Rice[6] obtained the short time near tip stress fields for a stationary crack in plane strain Mode I under step loading. The instantaneous response of the material is elastic. The initial asymptotic stress field as $r \to 0$ is therefore the elastic K field, i.e.

$$\sigma_{ij}(r \to 0, \theta, t = 0^+) = \frac{K_I}{\sqrt{2\pi r}} \hat{f}_{ij}(\theta), \tag{2.4}$$

where K_I is the stress intensity factor in Mode I, and the $\hat{f}_{ij}$ are normalized universal functions describing the angular variation of the stress components. For time $t > 0$ and $n > 1$, a new crack tip field develops within the region of dominance of the K field due to viscous relaxation. The K field is now regarded as the far field. The analytic form of this new stress field is similar to the well-known HRR singular field[7, 8], i.e.

$$\sigma_{ij}(r \to 0, \theta, t) = \left(\frac{C(t)}{BI_n r}\right)^{1/(n+1)} \tilde{\sigma}_{ij}(\theta, n), \tag{2.5}$$

where I_n is a numerical constant in the range 4–6 for plane strain, and the $\tilde{\sigma}_{ij}$s are normalized universal functions. The values of I_n and $\tilde{\sigma}_{ij}$ for different values of n and θ are given by Goldman and Hutchinson[9] and Shih[10]. For sufficiently small time, small-scale yielding conditions are satisfied, and creep deformation is confined to a region small compared to the region of dominance of the K field. An estimate of the creep zone boundary R_{cp} can be obtained by equating the equivalent creep strain ϵ_e^{cp} to the equivalent elastic strain ϵ_e^{el} and is [6]

$$R_{cp}(\theta, t) = \lambda_n K_I^2 (EBt)^{2/(n-1)} F_{cp}(\theta, n), \tag{2.6a}$$

$$\lambda_n = \frac{1}{2\pi}\left(\frac{(n+1)I_n}{2\pi(1-\nu^2)}\right)^{2/(n-1)}, \tag{2.6b}$$

where F_{cp} is a dimensionless function of order unity describing the angular extent of the creep zone. A more quantitative way of defining small-scale yielding is to demand R_{cp} to be much less than R_k, the region of dominance of the K field ($R_K \sim 0.1a_0$, where a_0 is the total crack length). The boundary of the region of asymptotic dominance of the HRR field described by (2.5) is given approximately by an expression similar to (2.6), except for the dimensionless function F_{cp}. As long as the small-scale yielding condition is satisfied, Riedel and Rice show that the amplitude $C(t)$ in (2.5) is a function of both K_I and time t:

$$C(t) = (1 - \nu^2)K_I^2/(n + 1)Et. \tag{2.7}$$

Equation (2.6) implies that small-scale yielding cannot persist for an unlimited amount of time in a specimen of finite size. For long times under constant applied loading, extensive creep occurs throughout the specimen. Under this circumstance $C(t)$ approaches the steady-state value of C^*. The transition time T between small-scale yielding and extensive creep can be estimated by setting $C(t)$ in (2.7) equal to C^*:

$$T = (1 - \nu^2)K_I^2/(n + 1)EC^*. \tag{2.8}$$

The above discussions show that, even for the simple case of a stationary crack under a suddenly applied constant load, the near tip fields cannot be uniquely specified by K_I but are also dependent on time. Furthermore, small-scale yielding is dependent on the loading history, time and material properties. The dependence of the near tip field on more than just a single parameter (such as C^* or K_I) should be regarded as the rule rather than the exception, particularly during transient crack growth.

Riedel[11] generalized the case of a suddenly applied load to the case when the load is applied gradually according to $P = P_0(t/t_0)^\beta$. P_0 and t_0 are reference load and time, respectively. As long as small-scale yielding persists, the stress intensity factor is also proportional to P_0, i.e.

$$K_I(t) = K_0(t/t_0)^\beta, \qquad K_0 \propto P_0. \tag{2.9}$$

Under small-scale yielding conditions, the stresses are self-similar and given by

$$\sigma = (1/EBt)^{1/(n-1)}\overline{\mathbf{F}}(\rho, \theta, n, \nu), \tag{2.10a}$$

$$\rho = r/(EBt)^{2/(n-1)}K_I^2(t). \tag{2.10b}$$

The near tip asymptotic stress field for time $t > 0$ is given by (2.5); the amplitude factor $C(t)$ is estimated to be

$$C(t) = (1 + 2\beta n)(1 - \nu^2)K_I^2(t)/(n + 1)Et. \tag{2.11}$$

Equations (2.5) and (2.10) imply that the asymptotic behavior of $\overline{\mathbf{F}}$ as $\rho \to 0$ is given by

$$\overline{F}_{ij}(\rho \to 0, \theta, n, \nu) = (\bar{c}_n/\rho)^{1/(n+1)}\tilde{\sigma}_{ij}(\theta, n), \tag{2.12a}$$

$$\bar{c}_n = (1 - \nu^2)(1 + 2\beta n)/(n + 1)I_n. \tag{2.12b}$$

For long times, $C(t)$ approaches $C^*(t)$, which is proportional to $t^{\beta(n+1)}$. Small-scale yielding conditions cannot be satisfied, and extensive creep occurs throughout the specimen. An estimate of the transition time T can be obtained by equating $C(t)$, given by (2.11), with $C^*(t)$. The resulting implicit equation for T is

$$C^*(T) = (1 - \nu^2)(1 + 2\beta n)K_I^2(T)/(n + 1)ET. \tag{2.13}$$

The creep zone boundary R_{cp} is estimated to be

$$R_{cp}(\theta, t) = \bar{\lambda}_n K_I^2(t)(EBt)^{2/(n-1)}F_{cp}(\theta, n), \tag{2.14a}$$

$$\bar{\lambda}_n = \frac{1}{2\pi}[2\pi\bar{c}_n]^{-2/(n-1)}, \tag{2.14b}$$

whereas the rate at which the creep zone spreads out ahead of the crack tip ($\theta = 0$) is given by

$$v_{cp}(t) = \gamma_n(EB)^{2/(n-1)}K_I^2(t)t^{-(n-3)/(n-1)} \tag{2.15a}$$

with

$$\gamma_n = 2(\beta + 1/(n - 1))\bar{\lambda}_n\overline{F}_{cp}, \tag{2.15b}$$

where $\overline{F}_{cp} = F(\theta = 0, n)$.

Growing cracks

For a quasi-statically growing crack with velocity $\dot{a}(t)$ along the positive x-axis, Hui and Riedel[12] have shown that a new type of singular field exists near the growing crack tip if $n > 3$. As $r \to 0$, the stresses in a plane strain Mode I growing crack are given by

$$\sigma_{ij}(r \to 0, \theta, n, \nu) = \alpha_n(\dot{a}/EBr)^{1/(n-1)}\hat{\sigma}_{ij}(\theta, n, \nu), \tag{2.16}$$

where α_n is a numerical constant dependent on n and ν, and the $\hat{\sigma}_{ij}$ are normalized dimensionless functions describing the angular variation of the stress components. The values of α_n and $\hat{\sigma}_{ij}$ for different values of n and θ are given in [12]. It is important to note that this singular field is valid for both steady and non-steady-state crack growth. Furthermore, the amplitude of this asymptotic field is completely determined by the current growth rate and is independent of the loading and the growth history. The region of dominance of this field, however, is generally dependent on the loading and growth history. Before the present work, this region of dominance was known only for limiting cases when the stress field is independent of the growth history (e.g. steady-state crack growth).

Hui[13] has considered the problem of steady-state crack growth under the small-scale yielding condition. For a plane strain Mode I growing crack under steady-state conditions, the stresses are given by

$$\sigma = (\dot{a}_{ss}/EBK_{\rm I}^2)^{1/(n-3)}\Sigma(R_1, \theta, n, \nu), \tag{2.17a}$$

$$R_1 = \frac{r}{[EBK_{\rm I}^{n-1}/\dot{a}_{ss}]^{2/(n-3)}}, \tag{2.17b}$$

where $\dot{a}_{ss}$ is the steady-state velocity, and Σ is a dimensionless tensor function of its dimensionless arguments. The small-scale yielding condition is satisfied by the requirement

$$\Sigma_{ij}(R_1 \to \infty, \theta, n, \nu) = (1/\sqrt{2\pi R_1})\hat{f}_{ij}(\theta). \tag{2.18}$$

The near tip asymptotic condition (2.16) is satisfied by the condition

$$\Sigma_{ij}(R_1 \to 0, \theta, n, \nu) = \alpha_n R_1^{-1/(n-1)}\hat{\sigma}_{ij}(\theta, n, \nu). \tag{2.19}$$

It should be noted that the steady-state crack growth problem under small-scale yielding conditions is correctly posed if $n > 3$[12, 13]. The region of dominance of the asymptotic field R_{HR} (2.16) is estimated to be a small fraction of the maximum creep zone extent L_{cp}, which is estimated to be

$$L_{cp} = l(n)(EBK_{\rm I}^{n-1}/\dot{a}_{ss})^{2/(n-3)}, \tag{2.20a}$$

$$l(n) = \frac{1}{4}\left[\frac{1}{\sqrt{2\pi}}\left(\frac{3}{2}\right)^{1/(n-1)}\right]^{2(n-1)/(n-3)} \tag{2.20b}$$

Equation (2.20a) implies that, for a finite specimen, $\dot{a}_{ss}$ must be sufficiently large to ensure small-scale yielding.

3. GOVERNING EQUATIONS

The governing equation of a growing crack with velocity $\dot{a}(t)$ will now be derived. We use a Cartesian coordinate system (x, y, z) with the z-axis lying along the crack front together with a polar coordinate system (r, θ). The origins of both coordinate systems move with the crack tip with velocity $\dot{a}(t)$ in the positive x-direction. We will only consider the case of a plane strain Mode I crack. The case of antiplane shear (Mode III) and plane stress Mode I follow in exactly the same manner. For mathematical convenience, we will present results for the case of an incompressible material, i.e. $\nu = 0.5$. As will be evident from the arguments in Section 5, this restriction can be removed and does not affect the qualitative nature of our solution.

The deformation behavior of an elastic power-law creeping material is given by (2.3). This material law is supplemented by the equilibrium equations

$$\nabla\cdot\boldsymbol{\sigma} = \mathbf{0} \tag{3.1}$$

and by the compatibility conditions, which, for plane strain, are

$$\nabla\cdot(\nabla\cdot\boldsymbol{\epsilon}') = \tfrac{2}{3}\nabla^2(\mathrm{tr}\,\boldsymbol{\epsilon}) - \nabla^2\epsilon_{33}, \tag{3.2}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\epsilon}'$ is the strain deviatoric tensor. The equilibrium equations (3.1) are satisfied by the introduction of the Airy stress function

$$\boldsymbol{\sigma} = -\nabla\nabla\phi + \mathbf{I}\nabla^2\phi. \tag{3.3}$$

The total time derivative, which has been denoted by a superposed dot in (2.3) is equal to

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} - \dot{a}(t)\frac{\partial}{\partial x} \tag{3.4}$$

in the moving coordinate system. Inserting the stress tensor according to (3.3) into the material law (2.3) and inserting the resulting strain rate tensor into the compatibility conditions (3.2), we have

$$\frac{2(1-\nu)}{E}\frac{\mathrm{D}}{\mathrm{D}t}(\nabla^2(\nabla^2\phi + S_{33})) - B\nabla\cdot(\nabla\cdot(\mathbf{I}(\nabla^2\phi - S_{33}) - 2\nabla\nabla\phi)\sigma_e^{n-1}) = 0, \tag{3.5}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\left(\frac{1-2\nu}{3}\right)\nabla^2\phi + S_{33}\right) + EB\sigma_e^{n-1}S_{33} = 0. \tag{3.6}$$

As in Riedel and Rice[6], the equivalent stress σ_e is given in terms of S_{33} and ϕ as

$$\sigma_e = (\sqrt{3}/2)(2(\nabla\nabla\phi:\nabla\nabla\phi) - (\nabla^2\phi)^2 + 3S_{33}^2)^{1/2}. \tag{3.7}$$

For incompressible materials ($\nu = .5$), eqns (3.5) and (3.6) are simplified since $S_{33} = 0$. In this case, the governing equation for transient crack growth reduces to

$$-\dot{a}(t)\frac{\partial}{\partial x}\nabla^4\phi + \frac{\partial}{\partial t}\nabla^4\phi - EB\nabla\cdot\{\nabla\cdot(I\nabla^2\phi - 2\nabla\nabla\phi)\sigma_e^{n-1}\} = 0. \tag{3.8}$$

The above equation can be obtained from the formulation of Riedel and Rice[6] simply by replacing the operator $\partial/\partial t$ by $\partial/\partial t - \dot{a}(t)(\partial/\partial x)$.

4. SELF-SIMILAR SOLUTION

In this section we will present a class of self-similar solutions for the transient problem of a plane strain Mode I growing crack under small-scale yielding conditions. The far field stress intensity factor is assumed to be proportional to t^{β}. The mechanics of transient crack growth are analyzed using these similarity solutions. In particular, the effect of growth history on small-scale yielding is examined, as well as the evolution of the asymptotic field found by Hui and Riedel. The approach to the limiting cases of "K controlled crack growth" and "C^* controlled crack growth" is presented using the similarity solutions.

The small-scale yielding assumption requires

$$\sigma_{ij}(r \to \infty, \theta, t) = (K_{\mathrm{I}}(t)/\sqrt{2\pi r})\tilde{f}_{ij}(\theta). \tag{4.1}$$

The stress intensity factor $K_I(t)$ is assumed to have the form

$$K_I(t) = K_0(t/t_0)^{\beta}. \tag{4.2}$$

Two special cases of interest, where the small-scale yielding assumption is expected to be satisfied, are

1. The creep zone is small compared to the specimen and the crack extension $a(t)$ is still smaller, i.e. $a(t) \ll R_{cp} \ll R_K$. This corresponds to a "slow" crack growth rate in the self-similar solution.
2. Crack extension $a(t)$ is large compared to the creep zone associated with the original crack (i.e. R_{cp}), but the creep zone near the growing crack is small compared to R_K, i.e. $R_{cp} \ll a(t)$. This situation corresponds to a "fast" crack growth rate in the self-similar solution.

We expect that transient effects (e.g. stress relaxation in the overall specimen) are important in the first case, whereas, for the case of large crack extension, relaxation effects are negligible. To quantify the above discussion, we introduce the condition of small crack extension

$$\dot{a}(t)/v_{cp}(t) = \mu(t) \ll 1, \tag{4.3}$$

where v_{cp} is given by (2.15a).

Note that, in general, the ratio of crack extension $a(t)$ to creep zone extent R_{cp} is a function of time t. In the self-similar solution $a(t)/R_{cp} \propto \dot{a}/v_{cp}$. This definition [i.e. eqn (4.3)] is motivated by the fact that, for small crack extension, the short time stationary result of Riedel [eqn (2.10)] is expected to remain valid for the growing crack problem, except for a very small zone near the moving crack tip, where the asymptotic result of Hui and Riedel must be satisfied. To investigate the growth of this asymptotic field, it is tempting to use (2.5) as the far field boundary condition

$$\sigma_{ij}(r \to \infty, \theta, t > 0) = (C(t)/BI_n r)^{1/(n+1)}\tilde{\sigma}_{ij}(\theta, n) \tag{4.4}$$

instead of the small-scale yielding condition (4.1). This boundary value problem, however, can lead to a contradiction if it is not posed carefully. To see why this is the case, consider each term of (3.8). The proposed far field boundary conditions require that, when the HRR field is applied to (3.8), the nonlinear third term must dominate all other terms in (2.7) as $r \to \infty$. For large r, the nonlinear term is of order $r^{-(n/(n+1))-2}$, the term $\nabla^4\phi_{,x}$ is of order $r^{-(1/(n+1))-3}$; and the term $\nabla^4\phi_{,t}$ is of order $r^{-(1/(n+1))-2}$. This implies that as $r \to \infty$, the $\nabla^4\phi_{,t}$ term dominates, which is a contradiction to our assumption. Note that the arguments used to arrive at this conclusion do not depend on self-similarity. In the correct formulation, the transient term is dropped in the outer boundary layer, and no contradiction occurs.

We will now show that a self-similar solution of (3.8) subject to the small-scale yielding condition is possible if the growth history is of the form

$$\dot{a}(t) = \dot{a}_0(t/t_0)^{\alpha}. \tag{4.5}$$

The constant α must satisfy the condition

$$\alpha = 2\beta - (n-3)/(n-1), \tag{4.6}$$

where $\dot{a}_0$ is a reference velocity.

The similarity condition can be understood physically as follows: Similarity requires the ratio of rate of crack advance to the rate of advance of the creep zone to be independent of time, i.e.

$$\mu(t) = \dot{a}(t)/v_{cp}(t) = \mu_0, \tag{4.7}$$

where μ_0 is a parameter independent of time. Using (2.15a) and (4.5), one can easily verify that (4.7) can be satisfied only if $\alpha = 2\beta - (n-3)/(n-1)$. Using the same equations, μ_0 is computed to be

$$\mu_0 = \frac{1}{\gamma_n}\left(\frac{\dot{a}_0 t_0}{[EBK_0^{n-1}/\dot{a}_0]^{2/(n-3)}}\right)^{(n-3)/(n-1)} \tag{4.8}$$

The similarity condition (4.6) implies that, for a steadily growing crack, the time variation of $K_I(t)$ must be proportional to $t^{(n-3)/2(n-1)}$. For the case of a suddenly applied constant load ($\beta = 0$), the similarity condition for a constantly growing crack is satisfied only by $n = 3$.

For general value of β, the small crack advance condition (4.3) is independent of time and is given by

$$\mu_0 \ll 1. \tag{4.9}$$

It is interesting to note that the factor $(EBK_0^{n-1}/\dot{a}_0)^{2/(n-3)}$ in (4.8) has dimensions of length and is related to the creep zone extent of a steadily growing crack under steady-state small-scale yielding conditions (2.20a).

To show that self-similar solutions exist if (4.6) is satisfied, let the stresses be given by

$$\boldsymbol{\sigma} = (1/EBt)^{1/(n-1)}\mathbf{H}(\rho, \theta, \mu_0, n), \tag{4.10a}$$

$$\rho = r/(EBt)^{2/(n-1)}K_I^2(t). \tag{4.10b}$$

The Airy stress function ϕ is then given by

$$\phi = (EBt)^{3/(n-1)}K_I^4(t)\Psi(\rho, \theta, \mu_0, n). \tag{4.11}$$

The condition of self-similarity implies that if (4.11) is substituted into (3.8) with $\dot{a}(t)$ given by (4.5), the function Ψ must satisfy an equation with ρ and θ as the only independent variables. This is indeed the case. A straightforward but tedious calculation shows that Ψ satisfies the partial differential equation

$$-\delta^{(n-3)/(n-1)}\bar{\nabla}^4\Psi_{,\bar{x}} + L(\Psi) + N(\Psi) = 0, \tag{4.12a}$$

where δ is defined by

$$\delta = (\gamma_n\mu_0)^{(n-1)/(n-3)} = \frac{\dot{a}_0 t_0}{(EBK_0^{n-1}/\dot{a}_0)^{2/(n-3)}}. \tag{4.12b}$$

L is the linear differential operator defined by

$$L(\Psi) = \left(4\beta + \frac{3}{n-1}\right)\bar{\nabla}^4\Psi - 2\left(\beta + \frac{1}{n-1}\right)\bar{\nabla}^4(\rho\Psi_{,\rho}). \tag{4.12c}$$

N is the nonlinear differential operator defined by

$$N(\Psi) = \sigma_e^{n-1}\bar{\nabla}^4\Psi + 4A_1(\Psi)A_1(\bar{\sigma}_e^{n-1}) + A_2(\Psi)A_2(\bar{\sigma}_e^{n-1}), \tag{4.12d}$$

where A_1 and A_2 are defined as

$$A_1 = \partial^2/\partial\bar{x}\partial\bar{y}, \tag{4.12e}$$

$$A_2 = \partial^2/\partial\bar{x}^2 - \partial^2/\partial\bar{y}^2, \tag{4.12f}$$

respectively. The dimensional variables x and y are nondimensionalized in exactly the same way as the variable $\rho = (\bar{x}^2 + \bar{y}^2)^{1/2}$. $\bar{\sigma}_e$ is the nondimensionalized equivalent stress defined by

$$\sigma_e = (1/EBt)^{1/(n-1)}\bar{\sigma}_e(\rho, \theta, \mu_0, n). \tag{4.12g}$$

The differential equation (4.12) must be supplemented by the far field boundary condition (4.1) and the usual traction free boundary conditions on the crack faces. Expressed in terms of the self-similar variable ρ, the far field boundary conditions are

$$\mathbf{H}(\rho \to \infty, \theta, \mu_0, n) = \hat{f}_{ij}(\theta)/\sqrt{2\pi\rho}. \tag{4.13}$$

By (2.20), the asymptotic behavior of $\mathbf{H}$ as $\rho \to 0$ is

$$\mathbf{H}(\rho \to 0, \theta, \mu_0, n) = \alpha_n \rho^{-1/(n-1)}\hat{\sigma}_{ij}(\theta, n). \tag{4.14}$$

Note that the first term of (4.12) corresponds to the elastic term $\dot{a}(t)\nabla^4\phi_{,x}$ of (3.8), the second term corresponds to the transient term $\nabla^4\phi_{,t}$, whereas the third term corresponds to the nonlinear term of (3.8). Note also that explicit dependence on the parameter μ_0 appears only in the first term of (4.12). If we set $\mu_0 = 0$ (e.g. $\dot{a}_0 = 0$), it can be easily verified that (4.12) reduces to the equations governing the self-similar stresses for a stationary crack subjected to the far field boundary conditions (4.1). Therefore, for any fixed ρ, as μ_0 approaches zero, we must recover the stationary self-similar solution or the outer solution

$$\mathbf{H}(\rho, \theta, \mu_0 \to 0, n, \nu) = \overline{\mathbf{F}}(\rho, \theta, n, \nu) = \mathbf{H}^{\text{out}}, \tag{4.15}$$

where $\mathbf{H}^{\text{out}}$ denotes the outer solution and $\overline{\mathbf{F}}$ is defined by (2.10a). Therefore, the solution (4.10) is a direct generalization of the self-similar stationary solution of Riedel and Rice[6].

We expect that the differential equation (4.12a) and boundary conditions (4.13) and (4.14) have solutions for Ψ for arbitrary μ_0. We are able to find explicit approximate results only for very large or very small values of μ_0.

For small values of μ_0 or $\delta \ll 1$, which corresponds to small crack extension, we anticipate that the stationary solution of Riedel and Rice dominates everywhere except for a small zone near the growing crack tip, inside which the asymptotic field of Hui and Riedel must be evolving. Mathematically, this is a typical singular perturbation problem, with $\delta^{(n-3)/(n-1)}$ as the small parameter multiplying the highest differentiated term (i.e. $\nabla^4\Psi_{,\bar{x}}$). The result of Hui and Riedel[12] indicates that any attempt to describe the stresses by a regular perturbation series will not be successful as the stresses are not analytic functions of δ near $\delta = 0$. Indeed, they show that the near tip asymptotic structure of the stress field is altered, as mentioned in Section 2 [eqn (2.16)]. On the other hand, it is plausible that in much of the region of interest the behavior of the stresses is governed by the stationary solution (2.10) or (3.15). The stresses given by eqns (2.10) or (4.15) are, for obvious reasons, defined as the outer solution of the boundary value problem [i.e. eqns (4.12)–(4.14)]. From the above discussions, we see that the problem of describing the evolution of the HR field as a function of the far field loading is mathematically equivalent to the problem of asymptotic matching. This problem will now be formulated.

The first step is to describe the boundary layer close to the crack tip (the region of dominance of the HR field). From the result of Hui and Riedel, we anticipate that the proper asymptotic balance in the boundary layer is between the first and third terms of (4.12). To see this, we define the following stretching variables:

$$\hat{\rho} = \rho/\delta^{((n+1)/2)(n-3)/(n-1)}, \qquad \hat{\Psi} = \Psi/\delta^{((2n+1)/2)(n-3)/(n-1)}. \tag{4.16}$$

Using the stretching variables, (4.12a) becomes

$$-\hat{\nabla}^4\hat{\Psi}_{,\hat{x}} + \delta^{(n-3)/2}L(\hat{\Psi}) + N(\hat{\Psi}) = 0, \tag{4.17}$$

where L and N are as previously defined by (4.12c)–(4.12f) but are written in terms of the stretched variables. The normalized stresses $\hat{\mathbf{H}}^{\text{in}}$ in the boundary layer are related to $\mathbf{H}$ by

$$\hat{\mathbf{H}}^{\text{in}} = \delta^{(n-3)/2(n-1)}\mathbf{H}.$$

Equation (4.17) is equivalent to the statement that, near the crack tip, the elastic strain and creep strain must balance. To first order, the transient term $L(\hat{\Psi})$ can be ignored. It is interesting to note that this is the term that caused problems when we attempted to formulate the boundary layer problem with the far field boundary conditions (4.4), as explained earlier in this section. In other words, "steady-state conditions" exist in the boundary layer formulation, in the sense that the transient term is much smaller than all other terms in (4.17). Thinking in terms of actual physical dimensions, this means that, in the self-similar solution, the transient terms are insignificant in a region whose size is considerably smaller than the creep zone. The governing equation for the stresses inside the boundary layer is therefore

$$-\hat{\nabla}^4\hat{\Psi}_{,\hat{x}} + N(\hat{\Psi}) = 0. \tag{4.18}$$

This equation must be supplemented by the usual traction free boundary conditions on the crack faces. The far field boundary condition is determined by the matching principle

$$\delta^{-(n-3)/2(n-1)}\hat{\mathbf{H}}^{\text{in}}(\hat{\rho} \to \infty, \theta, n) = \mathbf{H}^{\text{out}}(\rho \to 0, \theta, \delta, n) \sim \mathbf{F}(\rho \to 0, \theta, n), \tag{4.19}$$

where $\hat{\mathbf{H}}^{\text{in}}$ denotes the boundary layer solution. In (4.19), the rate at which $\rho \to 0$ must be less than $\delta^{((n+1)/2)(n-3)/(n-1)}$. This condition ensures that the transient term $\delta^{(n-3)/2}L(\hat{\Psi})$ is negligible compared with the rest of the terms in (4.17). Using (4.16) and the definition of stretching variables, the far field boundary condition for $\hat{\mathbf{H}}^{\text{in}}$ is

$$\hat{H}^{\text{in}}_{ij}(\hat{\rho} \to \infty, \theta, n) = (\bar{c}_n\hat{\rho})^{-1/(n+1)}\tilde{\sigma}_{ij}(\theta, n). \tag{4.20}$$

This far field boundary condition is exactly what we expect for small crack growth. The asymptotic behavior of $\hat{\mathbf{H}}^{\text{in}}$ as $\hat{\rho} \to 0$ is given by (2.20):

$$\hat{H}^{\text{in}}_{ij} = (\hat{\rho} \to 0, \theta, n) = \alpha_n\hat{\rho}^{-1/(n-1)}\hat{\sigma}_{ij}(\theta, n). \tag{4.21}$$

In general, the tensor function $\hat{\mathbf{H}}^{\text{in}}$ must be determined by numerical methods. However, it is possible to write a formula which interpolates the inner field (4.21) and the outer field (4.20):

$$\hat{H}^{\text{in}}_{ij} = \alpha_n\hat{\rho}^{-1/(n-1)}\hat{\sigma}_{ij}(\theta, n) + \bar{c}_n\hat{\rho}^{-1/(n+1)}\tilde{\sigma}_{ij}(\theta, n). \tag{4.22}$$

Note that the approximate formula above satisfies the near tip and far field boundary conditions [i.e. eqns (2.16) and (4.19)] asymptotically for small and large $\hat{\rho}$, respectively. The region of dominance of the HR field R_{HR} can be estimated by computing the equivalent stress of each of the terms in (4.22) and then setting them equal to each other. In terms of the variables r and θ, this is

$$R_{\text{HR}} = \delta^{((n+1)/2)(n-3)/(n-1)}h_n(EBt)^{2/(n-1)}K_I^2(t)F_c(\theta, n), \tag{4.23a}$$

$$h_n = \left(\frac{\alpha_n}{(\bar{c}_n)^{1/(n+1)}}\right)^{(n^2-1)/2}, \qquad F_c(\theta, n) = \left(\frac{\hat{\sigma}_e(\theta, n)}{\tilde{\sigma}_e(\theta, n)}\right)^{(n^2-1)/2}. \tag{4.23b}$$

R_{HR} is related to the creep zone extent R_{cp} by

$$R_{HR} = \delta^{((n+1)/2)(n-3)/(n-1)}\Lambda(\theta, n)R_{cp}(\theta, t), \tag{4.24a}$$

$$\Lambda(\theta, n) = (h_n/\bar{\lambda}_n)(F_c(\theta, n)/F_{cp}(\theta, n)), \tag{4.24b}$$

where $\bar{\lambda}_n$ and F_{cp} is defined by (2.14). R_{HR} is of the same order as the size of the region in which "steady-state" conditions are satisfied.

The above analysis indicates that as long as the small crack extension condition (4.9) is satisfied, the stationary analysis of Riedel and Rice can be applied almost everywhere except in the region defined by (4.24). Since this region is very small, the time of transition from small-scale yielding to extensive creep is still given by the result of the stationary analysis [i.e. (2.13)]. Notice also that, for small crack extension, three different asymptotic stress fields exist near the growing crack, one inside each other. They are given in the order of HR, HRR and K and are shown schematically in Fig. 1(a).

Consider now the other extreme where μ_0 is large. We would then expect creep relaxation or transient effects to be negligible everywhere. This extreme is shown schematically in Fig. 1(b). This case is in contrast with the previous case $\mu_0 \ll 1$, where transient effects or creep relaxation are negligible only in a very small region close to

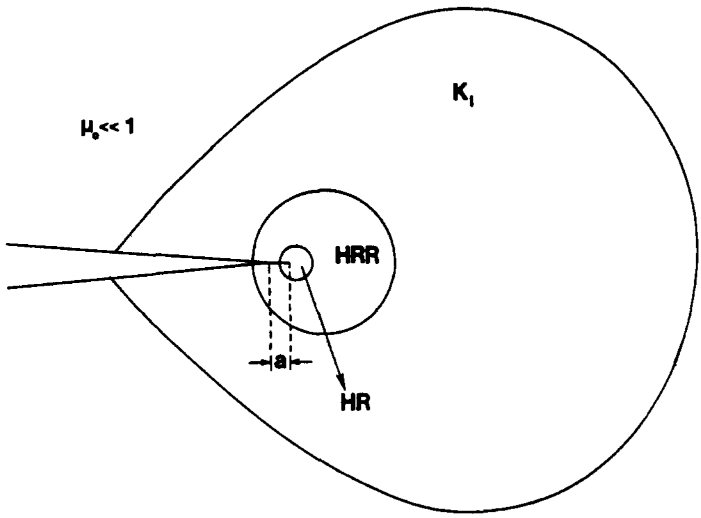

Fig. 1(a). Schematic figure showing the different regions of dominance of the asymptotic stress fields near the tip of a slowly growing crack in the self-similar solution.

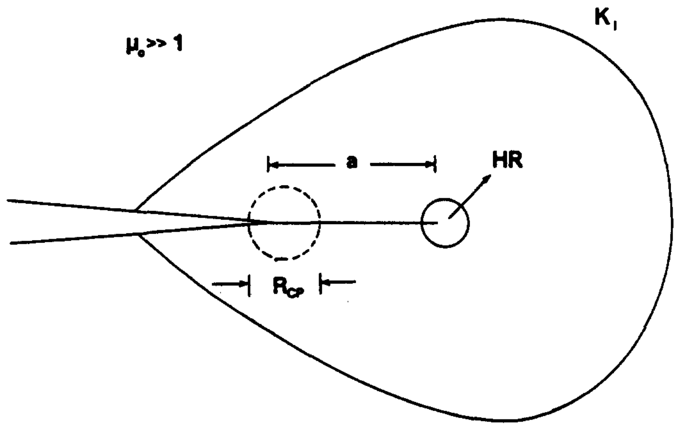

Fig. 1(b). Schematic figure showing the different regions of dominance of the asymptotic stress fields near the tip of a fast-growing crack in the self-similar solution. The dotted region represents the size of the creep zone if the crack were stationary.

the crack tip. To justify neglecting transient effects in the large μ_0 case, another form of the solution to the boundary value problem specified by (4.12)–(4.14) will be used. With a new choice of nondimensional variables the stresses can be expressed as

$$\sigma = [\dot{a}(t)/EBK_I^2(t)]^{1/(n-3)}\mathbf{Q}(\rho_1, \theta, \mu_0, n), \tag{4.25a}$$

where

$$\rho_1 = \frac{r}{(EBK_I^{n-1}(t)/\dot{a}(t))^{2/(n-3)}} \tag{4.25b}$$

and $\mathbf{Q}$ is a dimensionless tensor function of its dimensionless arguments. The nondimensionalized stress function Ψ_1 corresponding to (4.25) is

$$\phi = [EBK_I^2/\dot{a}(t)]^{3/(n-3)}K_I^4(t)\Psi_1(\rho_1, \theta, \mu_0, n). \tag{4.26}$$

The equation governing Ψ_1 can be obtained by substituting (4.26) into (3.8). The result is

$$-\tilde{\nabla}_1^4\Psi_{1,x} + (\mu_0\gamma_n)^{-(n-1)/(n-3)}L(\Psi_1) + N(\Psi_1) = 0. \tag{4.27}$$

The Cartesian coordinates (x_1, y_1) are nondimensionalized in exactly the same way as ρ_1 (4.25b), hence

$$\nabla_1^2 = \partial^2/\partial x_1 + \partial^2/\partial y_1^2.$$

Similar to the analysis of the $\mu_0 \ll 1$ case, the first term of (4.27) corresponds to the elastic term $\nabla^4\phi_{,x}$, the second term corresponds to the time transient term $\nabla^4\phi_{,t}$, whereas the last term corresponds to the nonlinear term in (3.8). Notice that the parameter μ_0 appears only with the transient term $L(\Psi_1)$. Comparing the magnitudes of each of the terms in (4.27), it is expected that the second term can be neglected due to the fact that $\mu_0 \gg 1$. Another interpretation of this approximation, is that the original field equation (3.8) becomes rate independent, i.e.

$$-\tilde{\nabla}_1^4\Psi_{1,x_1} + N(\Psi_1) = 0. \tag{4.28}$$

Since (4.28) does not contain μ_0 explicitly, we also expect Ψ_1 to be independent of μ_0. Thus, for large μ_0 we have

$$\sigma = [\dot{a}(t)/EBK_I^2(t)]^{1/(n-3)}\mathbf{Q}(\rho_1, \theta, n), \tag{4.29}$$

independent of μ_0.

Notice that (4.29) is identical in form to the steady-state solution (2.16). Both satisfy the same type of differential equation with similar boundary conditions. Thus, one might infer that sudden changes in crack extensions result in instantaneous changes in the stresses when the crack extensions and extension rates are large. Hui[13] has shown that the interpolation formulae suggested by Riedel provide a good approximation to the stresses in the steady-state growth problem. These formulae can be used to approximate $\mathbf{Q}$. The interpolation formula for each component of the stresses ahead of the crack tip is

$$\sigma_{ij} = \frac{\rho_1^{-1/2}}{\sqrt{2\pi}}\left(1 + \left(\frac{D_{ij}}{\rho_1}\right)^{(n-3)/2}\right)^{-1/(n-1)}, \quad i=j(\text{no sum}). \tag{4.30}$$

where D_{ij} is a numerical quantity defined in [13] to be the crossover value of the x-coordinate where the ij component of the far elastic field equals that of the asymptotic field (2.20) ahead of the crack tip. Using the interpolation formulae, the size of the region of dominance of the HR field ahead of the crack tip is estimated to have the form

$$R_{\mathrm{HR}} \propto (EBK_{\mathrm{I}}^2(t)/\dot{a}(t))^{2/(n-3)}. \tag{4.31}$$

The proportionality constant in (4.31) is of order D_{22} and is given in [13].

5. IMPLICATIONS FOR GENERAL GROWTH HISTORY

In this section, we will briefly outline how the results of the self-similar solution might be applied to the general case of arbitrary growth histories. The different values of the parameters μ_0 distinguish different stages of transient crack growth. Specifically, small values of $\mu_0 \ll 1$ [Figs 1(a) and 2(a)] imply that creep relaxation is important near the crack tip, except for a very small region near the crack tip, where the "steady-state condition" is dominant. In contrast, large values of μ_0 [Figs 1(b) and 2(b)] imply that "steady-state conditions" apply everywhere, and transient effects are negligible.

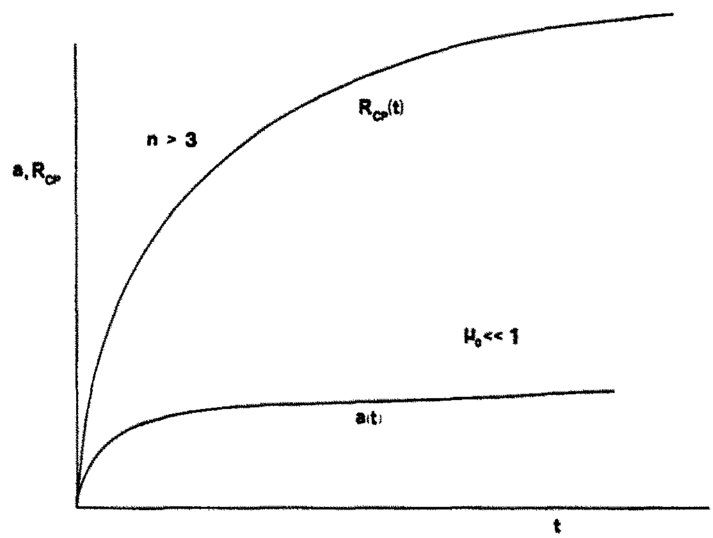

Fig. 2(a). Schematic drawing illustrating the definition of "slow" or "small crack extension" in the self-similar solution.

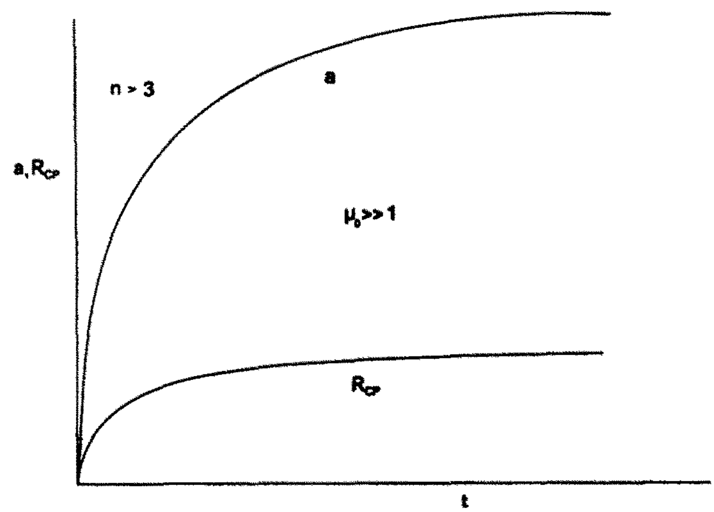

Fig. 2(b). Schematic drawing illustrating the definition of "fast" or "large crack extension" in the self-similar solution.

For general growth histories, μ_0 is time dependent and is redefined to be

$$\mu_0 \equiv \mu(t) = a(t)/R_{cp}(t), \tag{5.1}$$

where $a(t)$ is the net crack extension, and R_{cp} is the creep zone extent defined by (2.14). We anticipate that if $\mu(t)$ is uniformly small during the period of observation, then creep relaxation and transient effects are important. If the total time elapsed from the time of loading is less than the transition time of Rice and Riedel, the result of the stationary analysis [i.e. eqn (2.10)] should give an accurate description of the time-dependent stress fields except in a region significantly smaller than the creep zone. For example, a crack growing at constant rate (Fig. 3) probably has the same far HRR field as is given by small μ_0 for short time.

On the other hand, if $\mu_0(t)$ is much larger than one throughout most of the growth period, the transient term in eqn (3.8) is expected to be insignificant compared with the rest of the terms in (3.8) as long as changes in growth and loading rate satisfies the steady-state restrictions proposed by Riedel and Wagner[14]. The governing equation of crack growth reduces to

$$\dot{a}(t)\frac{\partial}{\partial x}\nabla^4\phi + EB\nabla\cdot\{(\nabla\cdot(\mathbf{I}\nabla^2\phi - 2\nabla\nabla\phi))\sigma_e^{n-1}\} = 0, \tag{5.2}$$

with the elastic K field as the far field boundary condition. Thus, "steady-state conditions" exist everywhere, and the stress field becomes rate independent. In this case, creep strain is everywhere small compared with elastic strain except very near the crack tip, where it must be of the same order. Results of previous steady-state crack growth analyses under small-scale yielding conditions[13] are expected to apply with $\dot{a}_{ss}$ and $\dot{a}(t)$ replaced by K_I and $K_I(t)$. For example, the stress field in the small-scale yielding problem has the form of eqn (2.17), with $\dot{a}_{ss}$ and K_I replaced by $\dot{a}(t)$ and $K_I(t)$. The dimensionless functions Σ are expected to be well approximated by the interpolation formula given in Hui[13], or eqn (4.30). These conditions are shown schematically for the case of constant growth rate in Fig. 3.

Finally, we will examine the evolution of the HR field for the case of small crack extension. For the same reasons as in the self-similar solution (Section 4), the transient terms of (3.9) should not be included in the outer boundary layer. If certain regularity conditions stated by Riedel and Wagner[14] are satisfied, we may be able to neglect

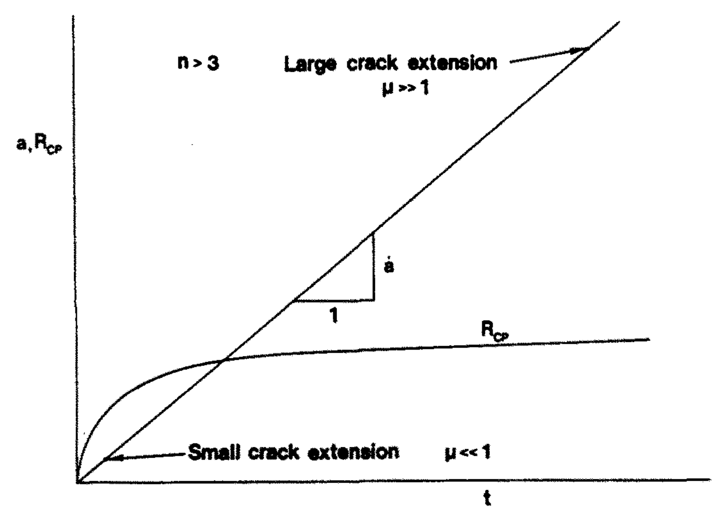

Fig. 3. Schematic drawing illustrating the definition of "small" and "large" crack extensions for a growing crack at constant rate. Small crack extension (or time) may correspond to $\mu_0 \ll 1$ in the self-similar solution. Large crack extension (or time) may correspond to $\mu_0 \gg 1$ in the self-similar solution.

transient terms in the inner boundary layer as well (Riedel, private communication). These regularity conditions imply limits on the magnitudes of the changes in crack growth rate and loading. Assuming these conditions are satisfied, the correct boundary layer formation is

$$\dot{a}(t)\frac{\partial}{\partial x}\nabla^4\phi + EB\nabla\cdot\{(\nabla\cdot(\mathbf{I}\nabla^2\phi - \nabla\nabla\phi))\sigma_e^{n-1}\} = 0, \tag{5.3a}$$

with the far field boundary condition

$$\sigma_{ij}(r \to \infty, \theta, t > 0) = (C(t)/BI_n r)^{1/(n+1)}\tilde{\sigma}_{ij}(\theta, n) \tag{5.3b}$$

and the usual traction free boundary conditions on the crack faces. As in (4.22), it is possible to write an approximate formula which interpolates the inner HR field and the outer HRR field (5.3b), i.e.

$$\sigma_{ij}(r, \theta, t) = \alpha_n\left(\frac{\dot{a}}{EBr}\right)^{1/(n-1)}\hat{\sigma}_{ij}(\theta, n) + \left(\frac{C(t)}{BI_n r}\right)^{1/(n+1)}\tilde{\sigma}_{ij}(\theta, n). \tag{5.4}$$

Similar approximate methods of matching the inner and outer stress fields have been used by Riedel and Wagner[14] and McClintock and Bassani[15]. Except for the special case of steady-state crack growth, i.e. (4.30), these interpolation formulae have not been checked by numerical results. The accuracy of these approximation formulae needs to be verified by future numerical computations on transient crack growth. The region of dominance of the HR field can be estimated by computing the equivalent stress of each of the terms in (5.4) and setting them equal to each other, i.e.

$$R_{\mathrm{HR}} = \alpha_n\left[\frac{\hat{\sigma}_e}{\tilde{\sigma}_e}\right]^{(n^2-1)/2}\left(\frac{\dot{a}(t)}{EB}\right)^{(n+1)/2}\left(\frac{BI_n}{C(t)}\right)^{(n-1)/2}. \tag{5.5}$$

For time greater than the transition time T, if the condition $\mu(t) \ll 1$ is satisfied, extensive creep will occur throughout most of the specimen. Elastic effects will be negligible practically everywhere, and C^* is the correct parameter to describe crack growth.

6. CONCLUSION AND DISCUSSION

Through the use of similarity solutions, we have examined the time evolution of the stress field near a growing crack tip. In the similarity solution, μ_0, the ratio of crack extension to the creep zone extent, is independent of time. Every value of μ_0 may correspond to a stage of crack growth in a non-self-similar solution. The similarity solution can be used to assess the accuracy of future numerical work.

The similarity solution allows us to make the following conclusions:

a. For short crack extension, i.e. $\mu_0 \ll 1$, the stationary result of Riedel and Rice[6] provides an excellent approximation to the crack growth problem. The concept of transition time of Riedel and Rice still applies as expected.
b. For short crack extension, we verified that the HR field, HRR field, and the K field coexist near the crack tip.
c. For fast crack growth characterized by $\mu_0 \gg 1$, creep relaxation can be neglected. The results of the steady-state analysis are expected to apply with very little modification.

Finally, we expect most features of these results to carry over to the general case when solutions are not self-similar. In this case, the parameter μ_0 defined by eqn (5.1) is time dependent. We expect that, if this parameter is uniformly small during the experiment, the stationary crack results apply. However, if it is large throughout most of the experiment, we can neglect relaxation effects (except for very small times after crack initiation).

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